COMPLETE SETS OF F-SQUARES FOR PRODUCTS OF PRIME NUMBERS.

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by

Walter T. Federer

ABSTRACT

Using a correspondence between multiplicative interactions and geometric interactions, a complete set of F-squares was obtained for squares of order arbitrary n. Some of the squares are modified F-squares rather than the usual ones. The complete set is not an orthogonal one although several squares are mutually pairwise orthogonal. The method is illustrated using n =6, n = 10, and n = 15.

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INTRODUCTION

The construction of complete sets of n $\times$ n F-squares for n equal to a power of a prime number is well-known and straight-forward. To date, the construction of complete sets of n $\times$ n F-squares for arbitrary n is unknown. We present a method herein for constructing complete sets of F-squares for n equal to a product of prime numbers, i.e, arbitrary n. The method is illustrated in detail for n = 6 and n = 10 and an indication of how the method works for n = 15. Some of the resulting squares are modified F-squares.

The number six, being a product of the two smallest prime numbers, has intrigued researchers at least since the time of L. Euler in the eighteenth century. Since a Latin square of order six has no orthogonal mate, an open question is "Does a complete set of F-squares for six exist?". Hedayat, Raghavarao, and Seiden (1975) found four orthogonal F-squares of three elements each repeated twice in a row and in a column, i. e., F(6; 3^2) or F(6; a^2, b^2, c^2). Using the results of the above authors, Anderson, Federer, and Seiden (1974) presented the following theorem which was used to construct eight F(6; a^2, b^2, c^2) squares:

Theorem: *Let n = 2t + 1. A set of r permutations of integers -t through t produces r - 1 orthogonal F(n+1, 2) squares if, when placed in an r $\times$ n array,*

(a) differences with the first row mod (n+1) reproduce -t through t, and

(b) differences of any other pair mod (t+1) produce 0 one time and 1, 2, ..., t each two times.

Federer (1975) with the aid of D. A. Anderson found an additional $F(6; a^3, b^3)$ square which when combined with one of the eight $F(6; a^2, b^2, c^2)$ squares formed a Latin square of order six and which was orthogonal to the remaining seven $F(6; a^2, b^2, c^2)$ squares. The set obtained was:

	Column					
Row	1	2	3	4	5	6
1	2011	1102	1221	0000	0212	2120
	0110	1110	2221	3221	4002	5002
2	2222	0110	1012	2101	0021	1200
	1021	0021	3102	2102	5210	4210
3	0110	2222	2101	1012	1200	0021
	2200	3200	4011	5011	0122	1122
4	0201	1020	2210	0122	1111	2002
	3012	2012	5120	4120	1201	0201
5	1020	0201	0122	2210	2002	1111
	4101	5101	0212	1212	2020	3020
6	1102	2011	0000	1221	2120	0212
	5222	4222	1000	0000	3111	2111

where the Latin square is in the first position in the second row of each group. The remaining positions are the seven $F(6; a^2, b^2, c^2)$ squares.

At the International Statistical Institute Meetings in Buenes Aires in 1981, the above results were discussed with D. J. Finney, who subsequently, among other things, obtained a similar result to the above (Finney, 1982). Some other work on partitions of the 6×6 square into sets with two symbols may be found in Federer (1982) and Johnson (1983a, 1983b).

Using the software package GENDEX, it was possible to computer generate ten orthogonal F-squares with three symbols and three orthogonal F-squares with two symbols. The output from this program was:

1	1	1	-1	0	0	-1	0	0	-1	1	-1	1
1	1	0	0	-1	1	-1	0	0	-1	-1	1	1
1	0	0	1	1	0	0	1	1	0	-1	-1	-1
1	0	-1	-1	-1	-1	0	1	1	0	1	1	-1
1	-1	-1	1	0	1	1	-1	-1	1	1	-1	1
1	-1	1	0	1	-1	1	-1	-1	1	-1	1	-1
1	1	1	1	1	1	-1	-1	1	0	1	1	1
1	1	-1	0	0	-1	-1	-1	1	0	-1	-1	-1

1	0	0	-1	0	1	0	0	-1	1	1	1	-1
1	0	1	0	-1	0	0	0	-1	1	-1	-1	1
1	-1	-1	-1	1	0	1	1	0	-1	-1	-1	-1
1	-1	0	1	-1	-1	1	1	0	-1	1	1	1
0	1	-1	0	0	-1	0	1	-1	-1	1	-1	1
0	1	1	1	1	1	0	1	-1	-1	-1	1	-1
0	0	1	0	-1	0	1	-1	0	0	-1	-1	-1
0	0	0	-1	0	1	1	-1	0	0	1	1	-1
0	-1	0	1	-1	-1	-1	0	1	1	-1	1	1
0	-1	-1	-1	1	0	-1	0	1	1	1	-1	1
0	1	-1	1	-1	0	0	-1	0	1	1	-1	-1
0	1	0	-1	1	-1	0	-1	0	1	-1	1	1
0	0	1	1	0	-1	1	0	1	-1	1	-1	-1
0	0	-1	0	1	1	1	0	1	-1	-1	1	1
0	-1	0	0	0	0	-1	1	-1	0	1	1	-1
0	-1	1	-1	-1	1	-1	1	-1	0	-1	-1	1
-1	1	0	-1	1	-1	1	0	-1	0	1	1	1
-1	1	-1	1	-1	0	1	0	-1	0	-1	-1	1
-1	0	-1	0	1	1	-1	1	0	1	-1	-1	-1
-1	0	1	1	0	-1	-1	1	0	1	1	1	-1
-1	-1	1	-1	-1	1	0	-1	1	-1	1	-1	-1
-1	-1	0	0	0	0	0	-1	1	-1	-1	1	1
-1	1	0	0	-1	1	1	1	1	1	-1	1	-1
-1	1	1	-1	0	0	1	1	1	1	1	-1	1
-1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1
-1	0	0	1	1	0	-1	-1	-1	-1	1	-1	-1
-1	-1	1	0	1	-1	0	0	0	0	-1	-1	1
-1	-1	-1	1	0	1	0	0	0	0	1	1	1

By replacing the -1 with a 2 in the first ten columns and the -1 with a 0 in the last three columns, the ten $F(6; a^2, b^2, c^2)$ squares and the three $F(6: a^3, b^3)$ squares result. It is not known if this program can produce more orthognal F-squares.

COMPLETE SETS OF F-SQUARES FOR N = 6

The six rows and six columns of a 6×6 lattice may be likened to $(2 \times 3) \times (2 \times 3)$ factorial. A partitioning the the degrees of freedom in this four factor factorial, A, B, C, and D, is given in Table 1. The interactions in a factorial are called multiplicative interactions whereas geometric interactions are needed for constructing F-squares. The geometric interactions corresponding to the multiplicative interactions are also given in Table 1. The geometric interactions are obtained by taking modulo the factor with the highest number of levels. Then using the levels of the geometric interactions to correspond to the numbers in an $F(6; a^2, b^2, c^2)$ square, a complete set of F-squares may be obtained. These are given in Table 2 with six symbols being given for factors A, B, and $A \times B$, the rows, and for C, D, and $C \times D$, the columns. There are twelve $F(6; a^2, b^2, c^2)$ squares and one $F(6: a^3, b^3)$

square which makes the set complete. Note that the four squares obtained from AD, ACD, BC, and ABC are not F-squares in both rows and columns. These squares all involve the two level factors A and C.

COMPLETE SETS OF F-SQUARES FOR N = 10

As with the square of order six, the square of order ten make be liked to a $(2 \times 5) \times (2 \times 5)$ four factor factorial. The partitioning of degrees of freedom for this $(2 \times 5) \times (2 \times 5)$ factorial with factors A, B, C, and D is given in Table 3. The corresponding geometrical interaction terms are included in the table. In addition to the rows and columns, there are twenty $F(10; a^2, b^2, c^2, d^2, e^2)$ squares and one $F(10; a^5, b^5)$ square which makes the set complete. The squares formed from AD, ACD, BC, and ABC are not F-squares in both rows and columns.

COMPLETE SETS OF F-SQUARES FOR N = 15

Likening the rows and columns of a 15 by 15 square to a four factor factorial, A, B, C, and D, with two of the factors at three levels and the other two with five levels each, we may partition the degrees of freedom as given in Table 5. The corresponding F-squares may be obtaining by taking the highest leveled factor in an interaction as was done for n = 6 and n = 10. This partitioning results in two $F(15; a^2, b^2, c^2)$ squares and 48 $F(15; a^2, b^2, c^2, d^2, e^2)$ squares.

COMMENTS

In an attempt to partition a square of order six into two $F(6; a^2, b^2, c^2)$ squares and one $F(6: a^3, b^3)$ consider the following:

A	000000	000000	000000	111111	111111	111111
B	001122	001122	001122	001122	001122	001122
A+B	001122	001122	001122	112200	112200	112200

Although the A+B square is orthogonal to the A square, it is not orthogonal to the B square. Each of the numbers occurs with itself six times but only occurs two times with each of the other numbers. Since A, B, and their multiplicative interaction are orthogonal, it is surprising that A, B, and their geometric interactions for non-primes are not. No way was found to associate the two degrees of freedom for a multiplicative interaction with the three symbols of an $F(6; a^2, b^2, c^2)$ square. The linear component of the multiplicative interaction could be associated with the comparison of symbol 0 with 2; the quadratic constrast did not appear to have any association with symbol 1 versus the other two in such a way that an $F(6; a^2, b^2, c^2)$ square orthogonal to the A and B squares could be constructed.

To partition the rows, say, of a square of order ten, consider the following:

A	0000000000	0000000000	0000000000	0000000000	0000000000
B	0011223344	0011223344	0011223344	0011223344	0011223344

A+B	0011223344	0011223344	0011223344	0011223344	0011223344
	1111111111	1111111111	1111111111	1111111111	1111111111
	0011223344	0011223344	0011223344	0011223344	0011223344
	1122334400	1122334400	1122334400	1122334400	1122334400

Although the F-square formed by A+B is orthogonal to the F(6: a^3, b^3) square, it is only in an incomplete block arrangement with the F-square formed by factor B.

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Table 1. ANOVA partitioning of degrees of freedom for a $2^2 \times 3^2$ factorial and related F-squares.

Source of variation	Degrees of freedom		Geometrical effects	Modulo
Correction for mean	1			
Rows	5			
A		1	A	2
B		2	B	3
$A \times B$		2	A+B	3
Columns	5			
C		1	C	2
D		2	D	3
$C \times D$		2	C+D	3
Treatments	5			
$A \times C$		1	A+C	2
$A \times D$		2	A+D	3
$A \times C \times D$		2	A+C+D	3
$B \times C$		2	B+C	3
$B \times D$	4			
BD		2	B+D	3
BD^2		2	$B+D^2$	3
$A \times B \times C$	2		A+B+C	3
$A \times B \times D$	4			
ABD		2	A+B+D	3
ABD^2		2	$A+B+D^2$	3
$A \times C \times D$	2		A+C+D	3
$B \times C \times D$	4			
BCD		2	B+C+D	3
BCD^2		2	$B+C+D^2$	3
$A \times B \times C \times D$	4			
ABCD		2	A+B+C+D	3
$ABCD^2$		2	$A+B+C+D^2$	3

Table 2. Array for a complete set of F-squares of order six.

row:	000000	111111	222222	333333	444444	555555
colum	012345	012345	012345	012345	012345	012345
AC	000111	000111	000111	111000	111000	111000
AD	000111	111222	222000	000111	111222	222000
ACD	000111	111222	222000	111222	222000	000111
BC	012012	012012	012012	120120	120120	120120
BD	012012	120120	201201	012012	120120	201201
BD^2	012012	201201	120120	012012	201201	120120
ABC	012120	012120	012120	120201	120201	120201
ABD	012120	120201	201012	012120	120201	201012
ABD^2	012120	201012	120201	012120	201012	120201
BCD	012012	120120	201201	120120	201201	012012
BCD^2	012012	201201	120120	120120	012012	201201
ABCD	012120	120201	201012	120201	201012	012120
$ABCD^2$	012120	201012	120201	120201	012120	201012

Table 3. ANOVA partitioning of degrees of freedom for a $2^2 \times 5^2$ factorial and related F-squares.

Source of variation	Degrees of freedom	Geometrical effects	Modulo
Correction for mean	1		
Rows	9		
A	1	A	2
B	4	B	5
$A \times B$	4	A+B	5
Columns	9		
C	1	C	2
D	4	D	5
$C \times D$	4	C+D	5
Treatments	9		
$A \times C$	1	A+C	2
$A \times D$	4	A+D	5
$A \times C \times D$	4	A+C+D	5
$B \times C$	4	B+C	5
$B \times D$	16		
BD	4	B+D	5
BD^2	4	$B+D^2$	5
BD^3	4	$B+D^3$	5

		BD^4	4	$B+D^4$	5
$A \times B \times C$	4			$A+B+C$	5
$A \times B \times D$	16				
		ABD	4	$A+B+D$	5
		ABD^2	4	$A+B+D^2$	5
		ABD^3	4	$A+B+D^3$	5
		ABD^4	4	$A+B+D^4$	5
$A \times C \times D$	4			$A+C+D$	3
$B \times C \times D$	16				
		BCD	4	$B+C+D$	5
		BCD^2	4	$B+C+D^2$	5
		BCD^3	4	$B+C+D^3$	5
		BCD^4	4	$B+C+D^4$	5
$A \times B \times C \times D$	16				
		$ABCD$	4	$A+B+C+D$	5
		$ABCD^2$	4	$A+B+C+D^2$	5
		$ABCD^3$	4	$A+B+C+D^3$	5
		$ABCD^4$	4	$A+B+C+D^4$	5

Table 4. Complete set of F-squares of order ten.

row:	0000000000	1111111111	2222222222	3333333333	4444444444	5555555555	6666666666	7777777777	8888888888	9999999999
col:	0123456789	0123456789	0123456789	0123456789	0123456789	0123456789	0123456789	0123456789	0123456789	0123456789
AC	0000011111	0000011111	0000011111	0000011111	0000011111	1111100000	1111100000	1111100000	1111100000	1111100000
AD	0000011111	1111122222	2222233333	3333344444	4444400000	0000011111	1111122222	2222233333	3333344444	4444400000
ACD	0000011111	1111122222	2222233333	3333344444	4444400000	1111122222	2222233333	3333344444	4444400000	0000011111
BC	0123401234	0123401234	0123401234	0123401234	0123401234	1234012340	1234012340	1234012340	1234012340	1234012340
BD	0123401234	1234012340	2340123401	3401234012	4012340123	0123401234	1234012340	2340123401	3401234012	4012340123
BD^2	0123401234	2340123401	4012340123	1234012340	3401234012	0123401234	2340123401	4012340123	1234012340	3401234012
BD^3	0123401234	3401234012	1234012340	4012340123	2340123401	0123401234	3401234012	1234012340	4012340123	2340123401
BD^4	0123401234	4012340123	3401234012	2340123401	1234012340	0123401234	4012340123	3401234012	2340123401	1234012340
BCD	0123401234	1234012340	2340123401	3401234012	4012340123	1234012340	2340123401	3401234012	4012340123	0123401234
BCD^2	0123401234	2340123401	4012340123	1234012340	3401234012	1234012340	3401234012	0123401234	2340123401	4012340123
BCD^3	0123401234	3401234012	1234012340	4012340123	2340123401	1234012340	4012340123	2340123401	0123401234	3401234012
BCD^4	0123401234	4012340123	3401234012	2340123401	1234012340	1234012340	0123401234	4012340123	3401234012	2340123401
ABC	0123412340	0123412340	0123412340	0123412340	0123412340	1234023401	1234023401	1234023401	1234023401	1234023401
ABD	0123412340	1234023401	2340134012	3401240123	4012301234	0123412340	1234023401	2340134012	3401240123	4012301234
ABD^2	0123412340	2340134012	4012301234	1234023401	3401240123	0123412340	2340134012	4012301234	1234023401	3401240123
ABD^3	0123412340	3401240123	1234023401	4012301234	2340134012	0123412340	3401240123	1234023401	4012301234	2340134012
ABD^4	0123412340	4012301234	3401240123	2340134012	1234023401	0123412340	4012301234	3401240123	2340134012	1234023401
ABCD	0123412340	1234023401	2340134012	3401240123	4012301234	1234023401	2340134012	3401240123	4012301234	0123412340
$ABCD^2$	01234123402340134012	40123012341234023401	3401240123	1234023401	3401240123	0123412340	2340134012	4012301234		
$ABCD^3$	01234123403401240123	12340234014012301234	2340134012	1234023401	4012301234	2340134012	0123412340	3401240123		
$ABCD^4$	01234123404012301234	34012401232340134012	1234023401	2340234012	1234023401	0123412340	4012301234	3401240123		

Table 5. ANOVA partitioning of degrees of freedom for a $3^2 \times 5^2$ factorial and related F-squares.

Source of variation	Degrees of freedom		Geometrical effects	Modulo
Correction for mean	1			
Rows	14			
A		2	A	3
B		4	B	5
A × B		8		
		4	A+B	5
		4	A+B^2	5
Columns	14			
C		2	C	3
D		4	D	5
C × D		8		
CD		4	C+D	5
CD^2		4	C+D^2	5
A × C	4			
AC		2	A+C	3
AC^2		2	A+C^2	3
A × D	8			
AD		4	A+D	5
AD^2		4	A+D^2	5
A × C × D	16			
ACD		4	A+C+D	5
ACD^2		4	A+C+D^2	5
ACD^3		4	A+C+D^3	5
ACD^4		4	A+C+D^4	5
B × C	8			
BC		4	B+C	5
BC^2		4	B+C^2	5
B × D	16			
BD		4	B+D	5
BD^2		4	B+D^2	5
BD^3		4	B+D^3	5
BD^4		4	B+D^4	5
A × B × C	16			
ABC		4	A+B+C	5
AB^2C		4	A+B^2+C	5
AB^3C		4	A+B^3+C	5
AB^4C		4	A+B^4+C	5
A × B × D	32			
ABD		4	A+B+D	5
ABD^2		4	A+B+D^2	5
ABD^3		4	A+B+D^3	5

		ABD^4	4	$A+B+D^4$	5
		AB^2D	4	$A+B^2+D$	5
		AB^2D^2	4	$A+B^2+D^2$	5
		AB^2D^3	4	$A+B^2+D^3$	5
		AB^2D^4	4	$A+B^2+D^4$	5
$A \times C \times D$	16				
		ACD	4	$A+C+D$	5
		ACD^2	4	$A+C+D^2$	5
		ACD^3	4	$A+C+D^3$	5
		ACD^4	4	$A+C+D^4$	5
$B \times C \times D$	32				
		BCD	4	$B+C+D$	5
		BCD^2	4	$B+C+D^2$	5
		BCD^3	4	$B+C+D^3$	5
		BCD^4	4	$B+C+D^4$	5
		BC^2D	4	$B+C^2+D$	5
		BC^2D^2	4	$B+C^2+D^2$	5
		BC^2D^3	4	$B+C^2+D^3$	5
		BC^2D^4	4	$B+C^2+D^4$	5
$A \times B \times C \times D$	64				
		$ABCD$	4	$A+B+C+D$	5
		$ABCD^2$	4	$A+B+C+D^2$	5
		$ABCD^3$	4	$A+B+C+D^3$	5
		$ABCD^4$	4	$A+B+C+D^4$	5
		AB^2CD	4	$A+B^2+C+D$	5
		AB^2CD^2	4	$A+B^2+C+D^2$	5
		AB^2CD^3	4	$A+B^2+C+D^3$	5
		AB^2CD^4	4	$A+B^2+C+D^4$	5
		AB^3CD	4	$A+B^3+C+D$	5
		AB^3CD^2	4	$A+B^3+C+D^2$	5
		AB^3CD^3	4	$A+B^3+C+D^3$	5
		AB^3CD^4	4	$A+B^3+C+D^4$	5
		AB^4CD	4	$A+B4+C+D$	5
		AB^4CD^2	4	$A+B^4+C+D^2$	5
		$AB4CD^3$	4	$A+B^4+C+D^3$	5
		AB^4CD^4	4	$A+B^4+C+D^4$	5